Unified Approach to the Derivation of Variational Expression for Electromagnetic Fields

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***Abstract*—A unified procedure to derive the variational expressions for electromagnetic field problems is presented. It is shown that the variational expressions for a variety of electromagnetic parameters such as, for instance, a resonant frequency, a propagation constant, and an impedance matrix, can be formulated systematically all from "the principle of least action" point of view. It is pointed out that the Maxwell's equations themselves can also be derived from the least action principle.**

I. Introduction

THE variational method is one of the very powerful techniques capable of treating a large variety of electromagnetic fields problems. So far, however, the appropriate variational expressions have been derived for each specific problem by a trial-and-error approach which is attended with laborious and time-consuming difficulties.

To avoid such a tricky approach, the procedures which are systematic to some extent have been proposed previously by several authors. Rumsey [1], Berk [2], and Harrington [3] have described the method to derive the variational expressions by using the "reaction concept," while Cairo and Kahan [4] have shown that the variational expressions can be obtained by means of "transpose operator and field." However, the method of "reaction concept" lacks uniformity in the strict sense because the definitions of reaction are different for the three-dimensional problems and the two-dimensional problems. In Cairo and Kahan's method, the physical meaning of the "transpose field" is hard to understand and also the technical skills are still required.

In the present paper, a unified procedure to derive the variational expressions for the electromagnetic field problems is given from "the principle of least action" point of view. We shall show first that the Maxwell's equations themselves can be yielded from the principle of least action, and then derive the variational expressions for a resonant frequency, a propagation constant, and an impedance matrix, systematically, all from the least action principle.

II. A Review of the Principle of Least Action

It is well known that a general feature of the classical physical phenomenon can be explained by the principle of least action, or, in other words, the stationarity of the actions which are suitably defined fundamental physical quantities such as energy, time, and others. The Fermat's principle in optics and the Thomson's theorem in the electrostatic fields are the typical examples. In dynamics, the system satisfies the Lagrange equation when the action, that is, the integration of Lagrange function with respect to time, is stationary. It follows that the principle of least action in dynamics states that "the motion of a matter passes through the path for which the action has the stationary value."

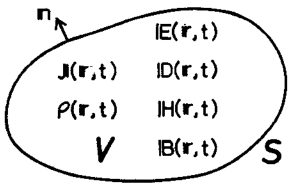

Fig. 1. Region V enclosed by closed surface S containing the electromagnetic fields, charge, and current. $\boldsymbol{n}$ is a unit vector normal to S and outwardly directed.

In electromagnetism, the action J is expressed in terms of Lagrange function L as follows [5]:

$$J = \int_{t_0}^{t_1} L \, dt \tag{1}$$

where

$$L = \int_V \left(\tfrac{1}{2}\{\boldsymbol{E}(\boldsymbol{r},t) \cdot \boldsymbol{D}(\boldsymbol{r},t) - \boldsymbol{H}(\boldsymbol{r},t) \cdot \boldsymbol{B}(\boldsymbol{r},t)\} + \boldsymbol{A}(\boldsymbol{r},t) \cdot \boldsymbol{J}(\boldsymbol{r},t) - \rho(\boldsymbol{r},t)\phi(\boldsymbol{r},t) \right) dv \tag{2}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}(\boldsymbol{r},t)$$

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} - \nabla\phi(\boldsymbol{r},t). \tag{3}$$

In the foregoing equation, $\boldsymbol{r}$ and t are the vector distance from the origin and the time, $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and the magnetic field intensities, $\boldsymbol{D}$ and $\boldsymbol{B}$ are the electric and the magnetic flux densities, $\boldsymbol{A}$ and ϕ are the vector and the scalar potentials, and $\boldsymbol{J}$ and ρ are the electric current density and the electric charge density, respectively. Then, the principle of least action for the electromagnetic fields can be stated as follows: "Provided that the correct values of $\boldsymbol{A}$ and ϕ are given in the region V at the initial time t_0 and the final time t_1 and on the closed surface S enclosing V during the time interval $[t_0,t_1]$, then $\boldsymbol{A}$ and ϕ in V for which J becomes stationary can be determined, and those $\boldsymbol{A}$ and ϕ thus determined give the true fields throughout the region V" (Fig. 1). The true values of $\boldsymbol{E}$ and $\boldsymbol{B}$ are then yielded from (3) using these $\boldsymbol{A}$ and ϕ, and further, the true expres-

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sions for $\boldsymbol{D}$ and $\boldsymbol{H}$ can be obtained with the aid of the constitutive relations.

Let us consider next another expression of the principle of least action for the electromagnetic fields in the frequency region. Extending the time interval $[t_0,t_1]$ to $(-\infty,\infty)$, the time domain problem of the electromagnetic fields can be transformed into the frequency domain problem by means of Fourier transformation. Thus the integration with respect to the time t in (1) becomes the integration with respect to the frequency f as follows:

$$J = \int_{-\infty}^{\infty} df \int_V (\tfrac{1}{2}\{\boldsymbol{E}(\boldsymbol{r},\omega)\cdot\boldsymbol{D}(\boldsymbol{r},\omega)^* - \boldsymbol{H}(\boldsymbol{r},\omega)\cdot\boldsymbol{B}(\boldsymbol{r},\omega)^*\} + \boldsymbol{A}(\boldsymbol{r},\omega)\cdot\boldsymbol{J}(\boldsymbol{r},\omega)^* - \rho(\boldsymbol{r},\omega)\phi(\boldsymbol{r},\omega)^*)\, dv \quad (4)$$

where * stands for the complex conjugate, $\boldsymbol{A}(\boldsymbol{r},\omega)$ and $\phi(\boldsymbol{r},\omega)$ are the Fourier transforms of $\boldsymbol{A}(\boldsymbol{r},t)$ and $\phi(\boldsymbol{r},t)$, respectively, and ω is an angular frequency. The frequency domain expressions of (3) are

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \nabla \times \boldsymbol{A}(\boldsymbol{r},\omega)$$
$$\boldsymbol{E}(\boldsymbol{r},\omega) = -j\omega\boldsymbol{A}(\boldsymbol{r},\omega) - \nabla\phi(\boldsymbol{r},\omega). \quad (5)$$

Let us assume that the materials involved are linear and dissipationless, for simplicity, but are inhomogeneous and anisotropic, in general. The constitutive relations can then be expressed in the form

$$\boldsymbol{D}(\boldsymbol{r},\omega) = \hat{\varepsilon}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega)$$
$$\boldsymbol{B}(\boldsymbol{r},\omega) = \hat{\mu}(\boldsymbol{r},\omega)\boldsymbol{H}(\boldsymbol{r},\omega). \quad (6)$$

Since the materials involved are assumed to be loss free, both the tensor permittivity $\hat{\varepsilon}$ and the tensor permeability $\hat{\mu}$ are Hermitian.

Since $\boldsymbol{E}(\boldsymbol{r},t)$, $\boldsymbol{H}(\boldsymbol{r},t)$, $\boldsymbol{D}(\boldsymbol{r},t)$, $\boldsymbol{B}(\boldsymbol{r},t)$, $\boldsymbol{J}(\boldsymbol{r},t)$, and $\rho(\boldsymbol{r},t)$ are real quantities, both $\boldsymbol{A}(\boldsymbol{r},t)$ and $\phi(\boldsymbol{r},t)$ are also the real functions. Therefore, the Fourier transformations of these real functions, $\boldsymbol{E}(\boldsymbol{r},\omega)$, $\boldsymbol{H}(\boldsymbol{r},\omega)$, $\boldsymbol{D}(\boldsymbol{r},\omega)$, $\boldsymbol{B}(\boldsymbol{r},\omega)$, $\boldsymbol{A}(\boldsymbol{r},\omega)$, $\phi(\boldsymbol{r},\omega)$, $\boldsymbol{J}(\boldsymbol{r},\omega)$, and $\rho(\boldsymbol{r},\omega)$, possess the following properties:

$$\boldsymbol{E}(\boldsymbol{r},\omega) = \boldsymbol{E}(\boldsymbol{r},-\omega)^*$$
$$\boldsymbol{H}(\boldsymbol{r},\omega) = \boldsymbol{H}(\boldsymbol{r},-\omega)^* \quad (7)$$

etc. Substituting the relations given by (6) and (7) into (4), we obtain

$$J = \int_0^{\infty} df \int_V (\boldsymbol{E}(\boldsymbol{r},\omega)^*\cdot\hat{\varepsilon}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) - \boldsymbol{H}(\boldsymbol{r},\omega)^* \cdot \hat{\mu}(\boldsymbol{r},\omega)\boldsymbol{H}(\boldsymbol{r},\omega) + \boldsymbol{A}(\boldsymbol{r},\omega)^*\cdot\boldsymbol{J}(\boldsymbol{r},\omega) + \boldsymbol{A}(\boldsymbol{r},\omega)\cdot\boldsymbol{J}(\boldsymbol{r},\omega)^* - \rho(\boldsymbol{r},\omega)\phi(\boldsymbol{r},\omega)^* - \rho(\boldsymbol{r},\omega)^*\phi(\boldsymbol{r},\omega))\, dv. \quad (8)$$

The first-order variation of the action J due to the small variation in ϕ is given as

$$\delta J_\phi = \int_0^{\infty} df \int_V \delta\phi^*(\nabla\cdot\hat{\varepsilon}\boldsymbol{E} - \rho)\, dv - \int_0^{\infty} df \int_{S_d} \delta\phi^*\hat{\varepsilon}\boldsymbol{E}\cdot\boldsymbol{n}\, ds + \text{c.c.} \quad (9)$$

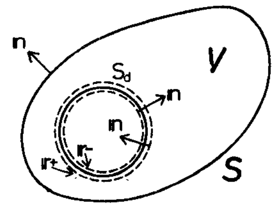

Fig. 2. Volume V enclosed by closed surface S. S_d indicates the surface across which $\hat{\varepsilon}$ and $\hat{\mu}$ change discontinuously. $\boldsymbol{r}^+$ and $\boldsymbol{r}^-$ designate the vector distances indicating outer and inner sides of discontinuous boundary S_d.

where the term designated as c.c. represents the complex conjugate of the first two terms on the right-hand side of the preceding equation. The surface integral must be carried out over both sides of the surface S_d across which the material constants $\hat{\varepsilon}$ and $\hat{\mu}$ change discontinuously. $\boldsymbol{n}$ is a unit vector normal to the boundary surface and directed as shown in Fig. 2.

In order that the action J be stationary for a variation in ϕ, the following equation

$$\nabla\cdot\hat{\varepsilon}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) = \rho(\boldsymbol{r},\omega) \quad (10)$$

must be satisfied in the region where $\hat{\varepsilon}$ and $\hat{\mu}$ are continuous, and at the same time, the following condition

$$\boldsymbol{n}\cdot\{\hat{\varepsilon}(\boldsymbol{r}^+,\omega)\boldsymbol{E}(\boldsymbol{r}^+,\omega) - \hat{\varepsilon}(\boldsymbol{r}^-,\omega)\boldsymbol{E}(\boldsymbol{r}^-,\omega)\} = 0 \quad (11)$$

must also be satisfied on both sides of discontinuity surface S_d. In the foregoing equation, $\boldsymbol{r}^+$ and $\boldsymbol{r}^-$ are the vector distance indicating outer and inner sides of the discontinuity surface S_d as shown in Fig. 2.

Similarly, the first-order variation of the action J with respect to $\boldsymbol{A}$ is given by

$$\delta J_A = \int_0^{\infty} df \int_V \delta\boldsymbol{A}^*\cdot(j\omega\hat{\varepsilon}\boldsymbol{E} - \nabla\times\boldsymbol{H} + \boldsymbol{J})\, dv + \int_0^{\infty} df \int_{S_d} (\boldsymbol{H}\times\delta\boldsymbol{A}^*)\cdot\boldsymbol{n}\, ds + \text{c.c.} \quad (12)$$

The notations in the preceding equation are the same as those used in (9). To make δJ_A zero for a variation in $\boldsymbol{A}$, we get

$$\nabla\times\boldsymbol{H}(\boldsymbol{r},\omega) = j\omega\hat{\varepsilon}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) + \boldsymbol{J}(\boldsymbol{r},\omega) \quad (13)$$

in the region where $\hat{\varepsilon}$ and $\hat{\mu}$ are continuous, and also

$$\boldsymbol{n}\times\{\boldsymbol{H}(\boldsymbol{r}^+,\omega) - \boldsymbol{H}(\boldsymbol{r}^-,\omega)\} = 0 \quad (14)$$

on both sides of discontinuity surface S_d. By choosing the trial functions for $\boldsymbol{A}$ and ϕ in such a way that $\boldsymbol{n}\times\boldsymbol{A}$, $\boldsymbol{n}\times\nabla\phi$, and ϕ are continuous across the discontinuity surface S_d, the following relation

$$\boldsymbol{n}\times\{\boldsymbol{E}(\boldsymbol{r}^+,\omega) - \boldsymbol{E}(\boldsymbol{r}^-,\omega)\} = 0 \quad (15)$$

is satisfied on both sides of S_d. Therefore, if $\boldsymbol{A}$ and ϕ are determined in such a manner that for those $\boldsymbol{A}$ and ϕ the action J becomes stationary and also $\boldsymbol{n}\times\boldsymbol{A}$, $\boldsymbol{n}\times\nabla\phi$, and ϕ are continuous across S_d, (5), (10), and (13), i.e., the Maxwell's equations, are satisfied in the region where $\hat{\varepsilon}$ and $\hat{\mu}$ are continuous, and at the same time, (14) and (15), i.e., the boundary conditions, are also satisfied on the discontinuity surface S_d.

We can conclude, therefore, that the physically realizable electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$, both satisfying the Maxwell's equations and the necessary boundary conditions, can be derived uniquely from the least action principle. It should be noted that the additional relation, that is, the so-called gauge condition, is required to determine the vector and scalar potentials $\boldsymbol{A}$ and ϕ uniquely.

III. Variational Expressions for Resonant Frequency

In this section, we shall derive the variational expressions for the resonant frequency of the cavity resonator using the results obtained in the preceding section.

We will assume that the cavity resonator is formed by a perfect conducting wall, and the materials involved in the resonator are inhomogeneous and anisotropic, in general. But the materials involved are assumed to be linear, nondispersive, and dissipation free, for simplicity. As a gauge condition, let ϕ be specified, for later convenience, as

$$\phi(r,\omega) = 0. \tag{16}$$

Under this condition, the relation between $\boldsymbol{A}$ and $\boldsymbol{E}$ given by (5) is reduced to

$$\boldsymbol{E}(r,\omega) = -j\omega \boldsymbol{A}(r,\omega). \tag{17}$$

We take for V in (8) the charge and current free region within the cavity resonator. Making use of (17), together with the aforementioned assumptions, reduces (8) to

$$J = \int_0^\infty df \int_V \Big(\boldsymbol{E}(r,\omega)^* \cdot \hat{\varepsilon}(r)\boldsymbol{E}(r,\omega) - \frac{1}{\omega^2}\{\nabla \times \boldsymbol{E}(r,\omega)\}^* \cdot \hat{\mu}^{-1}(r)\{\nabla \times \boldsymbol{E}(r,\omega)\}\Big)\, dv. \tag{18}$$

Thus the stationary problem being expressed in terms of $\boldsymbol{A}$ was transformed to the stationary problem expressed in terms of $\boldsymbol{E}$.

Since the electric field in a cavity resonator can be expressed by a linear combination of the electric fields of the individual resonant modes, $\boldsymbol{E}(r,\omega)$ can be written in the form

$$\boldsymbol{E}(r,\omega) = \sum_i \delta(\omega - \omega_i)\boldsymbol{E}_i(r) \tag{19}$$

where $\boldsymbol{E}_i(r)$ and ω_i are the electric field and the resonant frequency of the ith mode, respectively, and $\delta(\omega - \omega_i)$ signifies the delta function. Substituting (19) into (18), and carrying out the integration with respect to the frequency f, we get

$$J = \sum_i L_i \delta(0) \tag{20}$$

where

$$L_i = \int_V \Big(\boldsymbol{E}_i(r)^* \cdot \hat{\varepsilon}(r)\boldsymbol{E}_i(r) - \frac{1}{\omega_i^2}\{\nabla \times \boldsymbol{E}_i(r)\}^* \cdot \hat{\mu}^{-1}(r)\{\nabla \times \boldsymbol{E}_i(r)\}\Big)\, dv. \tag{21}$$

The amplitude of $\boldsymbol{E}_i(r)$ is determined by the electric field distributions at $t = -\infty$. If we assume that there exists only ith mode alone at $t = -\infty$, (20) becomes

$$J = L_i \delta(0). \tag{22}$$

Therefore, the stationary problem for J is reduced to that for L_i. Hence we can determine the correct values of the electric field $\boldsymbol{E}(r)$ and the resonant frequency ω in such a manner that for those $\boldsymbol{E}(r)$ and ω,

$$L = \int_V \Big(\boldsymbol{E}(r)^* \cdot \hat{\varepsilon}(r)\boldsymbol{E}(r) - \frac{1}{\omega^2}\{\nabla \times \boldsymbol{E}(r)\}^* \cdot \hat{\mu}^{-1}(r)\{\nabla \times \boldsymbol{E}(r)\}\Big)\, dv \tag{23}$$

becomes stationary. The subscript i will be omitted, for simplicity, hereafter.

The first integral on the right-hand side of (21) gives the electric energy, while the second represents the magnetic energy, both stored in the cavity resonator. Since the stored electric and the magnetic energies in the resonator are equal at the resonant frequency, L given by (23) must be zero for the correct value of the resonant frequency ω, which in turn leads to the expression [3]

$$\omega^2 = \frac{\int_V (\{\nabla \times \boldsymbol{E}(r)\}^* \cdot \hat{\mu}^{-1}(r)\{\nabla \times \boldsymbol{E}(r)\})\, dv}{\int_V (\boldsymbol{E}(r)^* \cdot \hat{\varepsilon}(r)\boldsymbol{E}(r))\, dv}. \tag{24}$$

In other words, the stationary problem for L given by (23) is equivalent to the stationary problem for ω^2 given by (24). We can conclude, therefore, that (24) represents the variational expression for the resonant frequency of the cavity resonator. The variational expression given by (24) coincides with that obtained previously by Berk [2].

The foregoing variational expression is expressed in terms of the electric field alone. Alternative to the preceding formulation, we may derive a similar expression which contains both electric and magnetic fields. By steps similar to those leading to (23) from (8), we get

$$L = \int_V (\{j\omega \boldsymbol{A}(r)\}^* \cdot \hat{\varepsilon}(r)\{j\omega \boldsymbol{A}(r)\} - \{\nabla \times \boldsymbol{A}(r)\}^* \cdot \hat{\mu}^{-1}(r)\{\nabla \times \boldsymbol{A}(r)\})\, dv \tag{25}$$

where

$$\boldsymbol{H}(r) = \hat{\mu}^{-1}(r)\nabla \times \boldsymbol{A}(r)$$
$$\boldsymbol{E}(r) = -j\omega \boldsymbol{A}(r). \tag{26}$$

Substituting the vector identity

$$(\nabla \times \boldsymbol{A})^* \cdot \hat{\mu}^{-1}(\nabla \times \boldsymbol{A}) = \boldsymbol{A}^* \cdot \nabla \times \hat{\mu}^{-1}\nabla \times \boldsymbol{A} - \nabla \cdot \{(\hat{\mu}^{-1}\nabla \times \boldsymbol{A}) \times \boldsymbol{A}^*\} \tag{27}$$

into (25) and applying Gauss' theorem, and further, in view of (26), we get the expression for L in the form

$$L = \int_V \Big(\boldsymbol{E}(r)^* \cdot \hat{\varepsilon}(r)\boldsymbol{E}(r) + \boldsymbol{H}(r)^* \cdot \hat{\mu}(r)\boldsymbol{H}(r) - \frac{j}{\omega}\{\boldsymbol{H}(r)^* \cdot \nabla \times \boldsymbol{E}(r) - \boldsymbol{E}(r)^* \cdot \nabla \times \boldsymbol{H}(r)\}\Big)\, dv - \frac{j}{\omega}\int_{S+S_d} \{\boldsymbol{H}(r) \times \boldsymbol{E}(r)^*\} \cdot \boldsymbol{n}\, ds \tag{28}$$

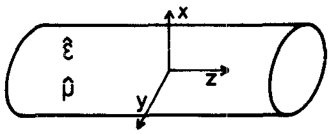

Fig. 3. Transmission line containing inhomogeneous and anisotropic materials. Direction of wave propagation is in z axis along which the line is uniform.

TABLE I
SYMMETRICAL TRANSFORMATIONS BETWEEN ELECTROMAGNETIC QUANTITIES

$\boldsymbol{E} \rightarrow \boldsymbol{H}$	$\hat{\varepsilon} \rightarrow \hat{\mu}$
$\boldsymbol{H} \rightarrow -\boldsymbol{E}$	$\hat{\mu} \rightarrow \hat{\varepsilon}$

where S indicates the surface of a perfect conducting wall enclosing a cavity resonator while S_d is the surface across which $\hat{\varepsilon}$ and $\hat{\mu}$ change discontinuously. Using the same procedure as that used to obtain (24), we found that

$$\omega = \frac{j\int_V \{\boldsymbol{H}(\boldsymbol{r})^* \cdot \nabla \times \boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{E}(\boldsymbol{r})^* \cdot \nabla \times \boldsymbol{H}(\boldsymbol{r})\}\, dv + j\int_{S+S_d} \{\boldsymbol{H}(\boldsymbol{r}) \times \boldsymbol{E}(\boldsymbol{r})^*\} \cdot \boldsymbol{n}\, ds}{\int_V \{\boldsymbol{E}(\boldsymbol{r})^* \cdot \hat{\varepsilon}(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{H}(\boldsymbol{r})^* \cdot \hat{\mu}(\boldsymbol{r})\boldsymbol{H}(\boldsymbol{r})\}\, dv}. \tag{29}$$

This is another variational expression for the resonant frequency of the cavity resonator which involves both electric and magnetic fields. It can be shown that the trial function for $\boldsymbol{E}$ in (29) must satisfy the conditions such that $\boldsymbol{n} \times \delta\boldsymbol{E} = 0$ on the conductor surface S and also $\boldsymbol{n} \times \delta\boldsymbol{E}$ is continuous across the discontinuity boundary S_d, where δ signifies a small variation in $\boldsymbol{E}$.

Making use of the symmetry of the Maxwell's equations shown in Table I, the variational expression (29) can be rewritten in slightly different form

$$\omega = \frac{j\int_V \{\boldsymbol{H}(\boldsymbol{r})^* \cdot \nabla \times \boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{E}(\boldsymbol{r})^* \cdot \nabla \times \boldsymbol{H}(\boldsymbol{r})\}\, dv - j\int_{S+S_d} \{\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r})^*\} \cdot \boldsymbol{n}\, ds}{\int_V \{\boldsymbol{E}(\boldsymbol{r})^* \cdot \hat{\varepsilon}(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{H}(\boldsymbol{r})^* \cdot \hat{\mu}(\boldsymbol{r})\boldsymbol{H}(\boldsymbol{r})\}\, dv} \tag{30}$$

which coincides with that obtained by Berk [2]. The trial function for $\boldsymbol{H}$ in (30) must satisfy the condition that $\boldsymbol{n} \times \delta\boldsymbol{H}$ is continuous across the discontinuity boundary S_d.

IV. VARIATIONAL EXPRESSIONS FOR PROPAGATION CONSTANT

Let us derive next the variational expressions for the propagation constant of the guided waves traveling along a uniform transmission line. It is assumed that the materials involved are inhomogeneous and anisotropic, in general, but are linear, nondispersive, and dissipation free. Fig. 3 illustrates the transmission line under consideration which is uniform in a direction of wave propagation z.

To derive the variational expressions for the propagation constant, divide the volume integral in (28) into the surface integral over the transverse (xy) plane S and the integral along the propagation axis z. Further, transforming the integration with respect to z into the integration with respect to the propagation constant β by performing the Fourier transformation, we get

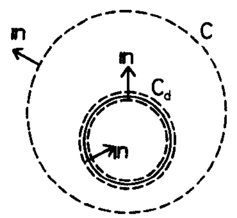

Fig. 4. Contours used in the evaluation of integration.

$$\begin{aligned} L = \frac{1}{2\pi\omega}\int_{-\infty}^{\infty} d\beta \Bigg[\int_S (&\{\omega \boldsymbol{E}(x,y,\beta)^* \cdot \hat{\varepsilon}(x,y)\boldsymbol{E}(x,y,\beta) \\ &+ \omega \boldsymbol{H}(x,y,\beta)^* \cdot \hat{\mu}(x,y)\boldsymbol{H}(x,y,\beta) \\ &- j\boldsymbol{H}(x,y,\beta)^* \cdot \nabla_t \times \boldsymbol{E}(x,y,\beta) \\ &+ j\boldsymbol{E}(x,y,\beta)^* \cdot \nabla_t \times \boldsymbol{H}(x,y,\beta)\} \\ &+ \beta\{\boldsymbol{H}(x,y,\beta)^* \cdot \boldsymbol{i}_z \times \boldsymbol{E}(x,y,\beta) \\ &- \boldsymbol{E}(x,y,\beta)^* \cdot \boldsymbol{i}_z \times \boldsymbol{H}(x,y,\beta)\})\, ds \\ &- j\int_{C+C_d} \{\boldsymbol{H}(x,y,\beta) \times \boldsymbol{E}(x,y,\beta)^*\} \cdot \boldsymbol{n}\, dl\Bigg]. \end{aligned} \tag{31}$$

In the foregoing equation, C represents a closed contour in the transverse plane as shown in Fig. 4. For the open-type transmission lines, C must be a closed contour enclosing the transmission line at infinity, while in the case of the metallic waveguide, C is a cross-sectional boundary of the guide wall. C_d indicates the line in the transverse plane across which $\hat{\varepsilon}$ and $\hat{\mu}$ change discontinuously. $\boldsymbol{E}(x,y,\beta)$ and $\boldsymbol{H}(x,y,\beta)$ are the Fourier transforms of $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$, respectively, and $\nabla_t = \boldsymbol{i}_x(\partial/\partial x) + \boldsymbol{i}_y(\partial/\partial y)$ where $\boldsymbol{i}_x$ and $\boldsymbol{i}_y$ are the unit vectors directed in x and y directions, respectively.

The electromagnetic fields of the wave propagation in the z direction can be expressed in terms of the linear combinations of the electromagnetic fields of each propagation mode. Hence

$$\begin{aligned} \boldsymbol{E}(x,y,\beta) &= \sum_i \delta(\beta + \beta_i)\boldsymbol{E}_i(x,y) \\ \boldsymbol{H}(x,y,\beta) &= \sum_i \delta(\beta + \beta_i)\boldsymbol{H}_i(x,y) \end{aligned} \tag{32}$$

where $\boldsymbol{E}_i(x,y)$ and $\boldsymbol{H}_i(x,y)$ represent the electric and the magnetic fields of the ith mode, respectively, and β_i is its propagation constant. Substituting (32) into (31), and carrying out the integration with respect to β, we get

$$L = \frac{1}{2\pi}\sum_i M_i \frac{\delta(0)}{\omega} \tag{33}$$

where

$$M_i = \int_S (\{\omega E_i(x,y)^* \cdot \hat{\varepsilon}(x,y)E_i(x,y) + \omega H_i(x,y)^* \cdot \hat{\mu}(x,y)H_i(x,y) - jH_i(x,y)^* \cdot \nabla_t \times E_i(x,y) + jE_i(x,y)^* \cdot \nabla_t \times H_i(x,y)\} - \beta_i\{H_i(x,y)^* \cdot i_z \times E_i(x,y) - E_i(x,y)^* \cdot i_z \times H_i(x,y)\})\, ds - j\int_{C+C_d} \{H_i(x,y) \times E_i(x,y)^*\} \cdot n\, dl. \tag{34}$$

The amplitude of the ith mode is determined by the electromagnetic fields at $t = -\infty$. If we assume that there exists only the ith mode alone at $t = -\infty$, the stationary problem for L is reduced to that for M_i, and hence we can determine the correct values of E_i, H_i, and β_i in such a way as M_i becomes stationary for those correct values. We shall neglect the subscript i for brevity hereafter. According to the similar reasoning used in the preceding section, M given by (34) must be zero for the correct values of E, H, and β (see Appendix). Therefore, the stationary problem for M is equivalent to the stationary problem for the propagation constant β given by

$$\beta = \frac{\int_S (\omega E(x,y)^* \cdot \hat{\varepsilon}(x,y)E(x,y) + \omega H(x,y)^* \cdot \hat{\mu}(x,y)H(x,y) - jH(x,y)^* \cdot \nabla_t \times E(x,y) + jE(x,y)^* \cdot \nabla_t \times H(x,y))\, ds - j\int_{C+C_d} \{H(x,y) \times E(x,y)^*\} \cdot n\, dl}{\int_S (H(x,y)^* \cdot i_z \times E(x,y) - E(x,y)^* \cdot i_z \times H(x,y))\, ds}. \tag{35}$$

The conditions to the trial function for E in (35) are $n \times \delta E = 0$ on C and $n \times \delta E$ changes continuously across C_d, where δ signifies a small variation in E. In view of the transformation given by Table I, (35) can be rewritten in the form

$$\beta = \frac{\int_S (\omega E(x,y)^* \cdot \hat{\varepsilon}(x,y)E(x,y) + \omega H(x,y)^* \cdot \hat{\mu}(x,y)H(x,y) - jH(x,y)^* \cdot \nabla_t \times E(x,y) + jE(x,y)^* \cdot \nabla_t \times H(x,y))\, ds + j\int_{C+C_d} \{E(x,y) \times H(x,y)^*\} \cdot n\, dl}{\int_S (H(x,y)^* \cdot i_z \times E(x,y) - E(x,y)^* \cdot i_z \times H(x,y))\, ds} \tag{36}$$

which coincides with that derived by Berk [2].[1] The trial function for H in (36) must satisfy the conditions $n \times \delta H = 0$ on C and $n \times \delta H$ changes continuously across C_d.

Provided that $\hat{\varepsilon}$ and $\hat{\mu}$ are in the form

$$\hat{\varepsilon}(x,y) = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{xy}^* & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \qquad \hat{\mu}(x,y) = \begin{pmatrix} \mu_{xx} & \mu_{xy} & 0 \\ \mu_{xy}^* & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{pmatrix} \tag{37}$$

the variational expression for the propagation constant can be expressed in terms of only the transverse component of the electric field alone, or only the transverse component of the magnetic field alone. According to the same procedure used to yield (31) from (28), we get from (21)

$$L = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\beta \int_S \Big(E(x,y,\beta)^* \cdot \hat{\varepsilon}(x,y)E(x,y,\beta) - \frac{1}{\omega^2}\{\nabla \times E(x,y,\beta)^* \cdot \hat{\mu}^{-1}(x,y)\nabla \times E(x,y,\beta)\}\Big)\, ds. \tag{38}$$

Dividing the electric field E into the transverse component e_t and the z component $i_z e_z$, and applying Gauss' law (10) with $\rho = 0$, we obtain

$$E(x,y,\beta) = e_t(x,y,\beta) + i_z e_z(x,y,\beta) \tag{39}$$

$$e_z(x,y,\beta) = -\frac{i_z \cdot \hat{\varepsilon}^{-1} i_z \nabla_t \cdot \hat{\varepsilon} e_t(x,y,\beta)}{j\beta}. \tag{40}$$

For the correct value of e_t the following equation, which is derived from the Maxwell's equations, must be satisfied:

$$-i_z \times \{-\beta^2 e_t(x,y,\beta) + \nabla_t i_z \cdot (\hat{\varepsilon}^{-1} i_z \nabla_t \cdot \hat{\varepsilon} e_t(x,y,\beta))\} = \omega^2 \hat{\mu}\{i_z \times \hat{\varepsilon} e_t(x,y,\beta)\} - \hat{\mu}\{i_z \times \nabla_t \times \hat{\mu}^{-1}\nabla_t \times e_t(x,y,\beta)\}. \tag{41}$$

Substituting (39)–(41) into (38), we get

$$L = \frac{1}{2\pi\omega^2}\int_{-\infty}^{\infty} d\beta \int_S \Big(\{\omega^2 e_t(x,y,\beta)^* \cdot \hat{\varepsilon} e_t(x,y,\beta) - (\nabla_t \times e_t(x,y,\beta)^*) \cdot \hat{\mu}^{-1}(\nabla_t \times e_t(x,y,\beta))\} - \frac{1}{\beta^2}[\{i_z \times (\omega^2 \hat{\varepsilon}^* e_t(x,y,\beta)^* - \nabla_t \times \hat{\mu}^{-1*}\nabla_t \times e_t(x,y,\beta)^*)\} \cdot \hat{\mu}\{i_z \times (\omega^2 \hat{\varepsilon} e_t(x,y,\beta) - \nabla_t \times \hat{\mu}^{-1}\nabla_t \times e_t(x,y,\beta))\} - \omega^2(i_z \nabla_t \cdot \hat{\varepsilon}^* e_t(x,y,\beta)^*) \cdot \hat{\varepsilon}^{-1}(i_z \nabla_t \cdot \hat{\varepsilon} e_t(x,y,\beta))]\Big)\, ds. \tag{42}$$

[1] Note that the last term's sign in the numerator of (14) in Berk's paper [2] is in error.

In the same way as that used to derive (35), we obtain from (42) the variational expression for β which is expressed in terms of the transverse electric field component $\boldsymbol{e}_t$ as follows:

$$\beta^2 = -\frac{\int_S (\{\boldsymbol{i}_z \times (\omega^2\hat{\varepsilon}^*\boldsymbol{e}_t^* - \nabla_t \times \hat{\mu}^{-1*}\nabla_t \times \boldsymbol{e}_t^*)\} \cdot \hat{\mu}\{\boldsymbol{i}_z \times (\omega^2\hat{\varepsilon}\boldsymbol{e}_t - \nabla_t \times \hat{\mu}^{-1}\nabla_t \times \boldsymbol{e}_t)\} - \omega^2(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}^*\boldsymbol{e}_t^*) \cdot \hat{\varepsilon}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}\boldsymbol{e}_t))\, ds}{\int_S ((\nabla_t \times \boldsymbol{e}_t^*) \cdot \hat{\mu}^{-1}(\nabla_t \times \boldsymbol{e}_t) - \omega^2\boldsymbol{e}_t^* \cdot \hat{\varepsilon}\boldsymbol{e}_t)\, ds}. \quad (43)$$

In the preceding equation, $\boldsymbol{e}_t = \boldsymbol{e}_t(x,y)$, $\hat{\varepsilon} = \hat{\varepsilon}(x,y)$, and $\hat{\mu} = \hat{\mu}(x,y)$ are the functions of x and y (independent of z). In particular, if materials involved are isotropic, $\hat{\varepsilon}$ and $\hat{\mu}$ in (43) become scalar constants, and (43) reduces to Kurokawa's variational expression [6] for the waveguide consists of perfectly conducting walls.

The first variation of (43) is given by

$$\delta\beta^2 \int_S \{(\nabla_t \times \boldsymbol{e}_t^*) \cdot \hat{\mu}^{-1}(\nabla_t \times \boldsymbol{e}_t) - \omega^2\boldsymbol{e}_t^* \cdot \hat{\varepsilon}\boldsymbol{e}_t\}\, ds$$

$$\begin{aligned}
&= -\int_S (\{\boldsymbol{i}_z \times (\nabla_t \times \hat{\mu}^{-1*}\nabla_t \times \delta\boldsymbol{e}_t^* - \omega^2\hat{\varepsilon}^*\delta\boldsymbol{e}_t^*)\}\\
&\quad \cdot \{\boldsymbol{i}_z \times (\beta^2\boldsymbol{e}_t - \nabla_t\boldsymbol{i}_z \cdot \hat{\varepsilon}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}\boldsymbol{e}_t)) - \hat{\mu}(\boldsymbol{i}_z \times (\omega^2\hat{\varepsilon}\boldsymbol{e}_t\\
&\quad - \nabla_t \times \hat{\mu}^{-1}\nabla_t \times \boldsymbol{e}_t))\})\, ds\\
&\quad - \beta^2 \int_{C+C_d} \{\boldsymbol{e}_t \times \hat{\mu}^{-1*}(\nabla_t \times \delta\boldsymbol{e}_t^*)\} \cdot \boldsymbol{n}\, dl\\
&\quad - \int_{C+C_d} ((\nabla_t \times \hat{\mu}^{-1*}\nabla_t \times \delta\boldsymbol{e}_t^* - \omega^2\hat{\varepsilon}^*\delta\boldsymbol{e}_t^*)\\
&\quad \cdot \{\boldsymbol{i}_z \cdot \hat{\varepsilon}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}\boldsymbol{e}_t)\}) \cdot \boldsymbol{n}\, dl + \text{c.c.} \qquad (44)
\end{aligned}$$

where C is a closed conductor in the transverse plane as in the case of (31), while C_d indicates the line in the transverse plane across which $\hat{\varepsilon}$, $\hat{\mu}$, and their first derivatives change discontinuously. The term designated as c.c. represents the complex conjugate of the terms before it. The correct value of $\boldsymbol{e}_t$ in (44) must satisfy the equation given by (41), and also both $\boldsymbol{n} \times \boldsymbol{e}_t$ and $\boldsymbol{i}_z \cdot \hat{\varepsilon}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}\boldsymbol{e}_t)$ must be continuous across C_d. Therefore, the conditions to the trial function for $\boldsymbol{e}_t$ in (43) can be stated as follows: 1) $\boldsymbol{n} \times \hat{\mu}^{-1}\nabla_t \times \delta\boldsymbol{e}_t$ and $\boldsymbol{n} \cdot (\nabla_t \times \hat{\mu}^{-1}\nabla_t \times \delta\boldsymbol{e}_t - \omega^2\hat{\varepsilon}\delta\boldsymbol{e}_t)$ must vanish on C; 2) $\boldsymbol{n} \times \hat{\mu}^{-1}\nabla_t \times \delta\boldsymbol{e}_t$ and $\boldsymbol{n} \cdot (\nabla_t \times \hat{\mu}^{-1}\nabla_t \times \delta\boldsymbol{e}_t - \omega^2\hat{\varepsilon}\delta\boldsymbol{e}_t)$ must change continuously across C_d. In order to simplify the conditions to the trial function for $\boldsymbol{e}_t$, let us add the term of line integral to both the denominator and the numerator in (43) as follows:

$$\beta^2 = -\frac{\begin{aligned}&\int_S (\{\boldsymbol{i}_z \times (\omega^2\hat{\varepsilon}^*\boldsymbol{e}_t^* - \nabla_t \times \hat{\mu}^{-1*}\nabla_t \times \boldsymbol{e}_t^*)\} \cdot \hat{\mu}\{\boldsymbol{i}_z \times (\omega^2\hat{\varepsilon}\boldsymbol{e}_t - \nabla_t \times \hat{\mu}^{-1}\nabla_t \times \boldsymbol{e}_t)\}\\ &- \omega^2(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}^*\boldsymbol{e}_t^*) \cdot \hat{\varepsilon}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}\boldsymbol{e}_t))\, ds - \int_{C+C_d} ((\nabla_t \times \hat{\mu}^{-1*}\nabla_t \times \boldsymbol{e}_t^* - \omega^2\hat{\varepsilon}^*\boldsymbol{e}_t^*)\{\boldsymbol{i}_z \cdot \hat{\varepsilon}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}\boldsymbol{e}_t)\}\\ &+ (\nabla_t \times \hat{\mu}^{-1}\nabla_t \times \boldsymbol{e}_t - \omega^2\hat{\varepsilon}\boldsymbol{e}_t)\{\boldsymbol{i}_z \cdot \hat{\varepsilon}^{-1*}(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}^*\boldsymbol{e}_t^*)\}) \cdot \boldsymbol{n}\, dl\end{aligned}}{\int_S ((\nabla_t \times \boldsymbol{e}_t^*) \cdot \hat{\mu}^{-1}(\nabla_t \times \boldsymbol{e}_t) - \omega^2\boldsymbol{e}_t^* \cdot \hat{\varepsilon}\boldsymbol{e}_t)\, ds - \int_{C+C_d} (\boldsymbol{e}_t \times \hat{\mu}^{-1*}(\nabla_t \times \boldsymbol{e}_t^*) + \boldsymbol{e}_t^* \times \hat{\mu}^{-1}(\nabla_t \times \boldsymbol{e}_t)) \cdot \boldsymbol{n}\, dl}. \quad (45)$$

Then the conditions to the trial function for $\boldsymbol{e}_t$ derived from the first variation of (45) become in simpler form as follows: 1) $\boldsymbol{n} \times \delta\boldsymbol{e}_t$ and $\boldsymbol{i}_z \cdot \hat{\varepsilon}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}\delta\boldsymbol{e}_t)$ must vanish on C; 2) $\boldsymbol{n} \times \delta\boldsymbol{e}_t$ and $\boldsymbol{i}_z \cdot \hat{\varepsilon}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\varepsilon}\delta\boldsymbol{e}_t)$ must change continuously across C_d.

The variational expression for the propagation constant expressed in terms of transverse magnetic field component $\boldsymbol{h}_t$ can easily be obtained from (45) with the aid of the transformation given by Table I. The result is

$$\beta^2 = -\frac{\begin{aligned}&\int_S (\{\boldsymbol{i}_z \times (\omega^2\hat{\mu}^*\boldsymbol{h}_t^* - \nabla_t \times \hat{\varepsilon}^{-1*}\nabla_t \times \boldsymbol{h}_t^*)\} \cdot \hat{\varepsilon}\{\boldsymbol{i}_z \times (\omega^2\hat{\mu}\boldsymbol{h}_t - \nabla_t \times \hat{\varepsilon}^{-1}\nabla_t \times \boldsymbol{h}_t)\}\\ &- \omega^2(\boldsymbol{i}_z\nabla_t \cdot \hat{\mu}^*\boldsymbol{h}_t^*) \cdot \hat{\mu}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\mu}\boldsymbol{h}_t))\, ds - \int_{C+C_d} ((\nabla_t \times \hat{\varepsilon}^{-1*}\nabla_t \times \boldsymbol{h}_t^* - \omega^2\hat{\mu}^*\boldsymbol{h}_t^*)\{\boldsymbol{i}_z \cdot \hat{\mu}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\mu}\boldsymbol{h}_t)\}\\ &+ (\nabla_t \times \hat{\varepsilon}^{-1}\nabla_t \times \boldsymbol{h}_t - \omega^2\hat{\mu}\boldsymbol{h}_t)\{\boldsymbol{i}_z \cdot \hat{\mu}^{-1*}(\boldsymbol{i}_z\nabla_t \cdot \hat{\mu}^*\boldsymbol{h}_t^*)\}) \cdot \boldsymbol{n}\, dl\end{aligned}}{\int_S ((\nabla_t \times \boldsymbol{h}_t^*) \cdot \hat{\varepsilon}^{-1}(\nabla_t \times \boldsymbol{h}_t) - \omega^2\boldsymbol{h}_t^* \cdot \hat{\mu}\boldsymbol{h}_t)\, ds - \int_{C+C_d} (\boldsymbol{h}_t \times \hat{\varepsilon}^{-1*}(\nabla_t \times \boldsymbol{h}_t^*) + \boldsymbol{h}_t^* \times \hat{\varepsilon}^{-1}(\nabla_t \times \boldsymbol{h}_t)) \cdot \boldsymbol{n}\, dl}. \quad (46)$$

The conditions to the trial function for $\boldsymbol{h}_t$ in (46) can be found from the first variation of (46). Those conditions are as follows: 1) $\boldsymbol{n} \times \delta\boldsymbol{h}_t$ and $\boldsymbol{i}_z \cdot \hat{\mu}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\mu}\delta\boldsymbol{h}_t)$ must vanish on C; 2) $\boldsymbol{n} \times \delta\boldsymbol{h}_t$ and $\boldsymbol{i}_z \cdot \hat{\mu}^{-1}(\boldsymbol{i}_z\nabla_t \cdot \hat{\mu}\delta\boldsymbol{h}_t)$ must change continuously across C_d.

V. Variational Expression for Impedance Matrix

Let us derive in this section the variational expression for the impedance matrix of waveguide junctions from the least action principle. The n-port waveguide junction under consideration is shown schematically in Fig. 5. Suppose that the materials involved in the junction are linear, nondispersive, and dissipation free. In Fig. 5, S_i $(i = 1,2,\cdots,n)$ are the reference planes chosen in the connected waveguides far enough from the junction so that all the higher modes above cutoff are extinguished. Let the electric and magnetic fields within the junction be $\boldsymbol{E}_i$ and $\boldsymbol{H}_i$, respectively, when all ports except the ith port are open circuited and a unit current is fed into the ith port. Similarly, $\boldsymbol{E}_j$ and $\boldsymbol{H}_j$ may be regarded as the electric and magnetic fields within the junction caused by an input unit current at port j when all ports except the jth port are open circuited.

Because of the linearity of the system under consideration, the electromagnetic fields within the junction corresponding to the unit input currents at ports i and j can then be expressed as

$$\boldsymbol{E} = \boldsymbol{E}_i + \boldsymbol{E}_j$$
$$\boldsymbol{H} = \boldsymbol{H}_i + \boldsymbol{H}_j. \quad (47)$$

We take for V in (8) the charge and current free region within the junction, and assume that the junction is operated

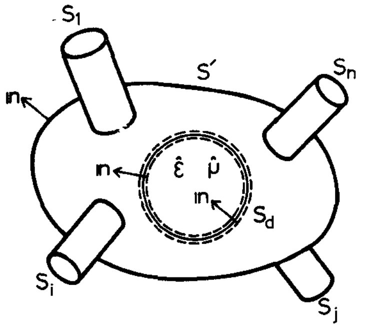

Fig. 5. Schematic diagram of n-port waveguide junction.

with single angular frequency ω. Then (8) becomes as

$$J = \delta(0) \int_V (E(r)^* \cdot \hat{\varepsilon}(r)E(r) - H(r)^* \cdot \hat{\mu}(r)H(r))\, dv$$
$$= -\frac{\delta(0)}{j\omega} \sum_{i,j} Z_{ij} \tag{48}$$

where Z_{ij} is a component of an impedance matrix given by

$$Z_{ij} = j\omega \int_V (H_i^* \cdot \hat{\mu} H_j - E_i^* \cdot \hat{\varepsilon} E_j)\, dv. \tag{49}$$

Since both E and H in the foregoing equation are the functions of A and ϕ, we can rewrite (49), with the aid of (5) and (16), in the form

$$Z_{ij} = j\omega \int_V \left\{H_i^* \cdot \hat{\mu} H_j - \frac{1}{\omega^2} (\nabla \times H_i^*) \cdot \hat{\varepsilon}^{-1} (\nabla \times H_j)\right\} dv. \tag{50}$$

Thus the stationary problem for J given by (48) has been reduced to the stationary problem for Z_{ij} given by (50) expressed in terms of H.

The first-order variation of Z_{ij} is given by

$$\delta Z_{ij} = \int_V \{\delta H_i^* \cdot (\nabla \times E_j + j\omega \hat{\mu} H_j) + \delta H_j \cdot (-\nabla \times E_i^* + j\omega \hat{\mu}^* H_i^*)\}\, dv + \int_{S' + S_d + S_r} \{\delta H_i^* \times E_j - \delta H_j \times E_i^*\} \cdot n\, ds \tag{51}$$

where S' is a conductor wall forming a junction, S_d is a surface across which $\hat{\varepsilon}$ and $\hat{\mu}$ change discontinuously, and S_r are the reference planes in the connected waveguides, as shown in Fig. 5. The conditions to the trial function for H in (50) can then be found from (51) as follows: 1) $n \times \delta H = 0$ on S_r; 2) $n \times \delta H$ changes continuously across S_d.

VI. Conclusion

A novel approach to derive the variational expressions for electromagnetic field problems has been proposed. It has been shown that the variational expressions for a resonant frequency of a cavity resonator, a propagation constant of a uniform transmission line, and an impedance matrix of a waveguide junction can be derived systematically all from the least action principle. We have shown that the Maxwell's equations themselves can also be yielded by solving the stationary problem of the action J.

Though we have assumed the materials contained in the system under consideration are linear and loss free, the variational expressions derived in the present paper are applicable to the system which involves anisotropic and/or inhomogeneous materials.

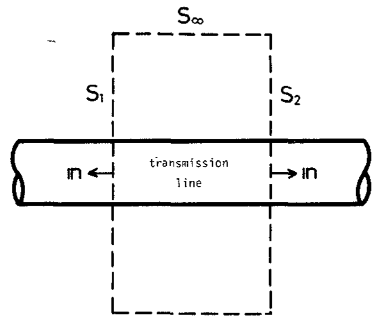

Fig. 6. Surface $S(S_1 + S_2 + S_\infty)$ used in evaluation of integration in (A3). S_∞ is a cylindrical side surface with infinite radius, and S_1 and S_2 are parallel surfaces transverse to the propagation direction of the transmission line.

Appendix

For the charge and the current free region V, (8) is reduced to

$$J = \int_0^\infty df \int_V (E(r,\omega)^* \cdot \hat{\varepsilon}(r,\omega)E(r,\omega) - H(r,\omega)^* \cdot \hat{\mu}(r,\omega)H(r,\omega))\, dv. \tag{A1}$$

Provided that E and H are the correct fields satisfying the Maxwell's equations for the loss-free region, the following equation

$$\nabla \cdot (E \times H^*) = j\omega(E^* \cdot \hat{\varepsilon} E - H^* \cdot \hat{\mu} H) \tag{A2}$$

is derived. Substituting (A2) into (A1) and applying Gauss' theorem, reduce (A1) to

$$J = \int_0^\infty df \int_S \left(\frac{1}{j\omega} E \times H^*\right) \cdot n\, ds. \tag{A3}$$

As shown in Fig. 6, the surface S consists of S_1, S_2, and S_∞. J given by (A3) vanishes for a propagation mode because $(E \times H^*) \cdot n$ is zero on S_∞ and also the unit normal vectors n on S_1 and S_2 direct to opposite directions. Therefore, the action J, and hence L, becomes zero for a propagation mode. We can conclude, therefore, that M must be zero for a correct propagation mode.

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